Algebra II — Lesson 0509

Dr. Paul L. Bailey — Wednesday, May 11, 2022

## 1. Review

Recall our definitions regarding probability. We perform some sort of "experiment", such as rolling a die. An *outcome* is result of the experiment; in this case, the outcome is an integer between one and six. The *sample space* is the set of all possible outcomes. An *event* is a subset of the sample space.

The *cardinality* of a set is the number of things in it. The cardinality of the set $A$ is denoted $|A|$.

The *probability* of event $E$ is defined to be

$$P(E) = \frac{|E|}{|S|}.$$

For a positive integer $n$, we define $n$ factorial as the product of positive integers less than or equal to $n$:

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n.$$

This is the number of ways of rearranging $n$ things.

The number of *permutations* (ordered subsets) of $n$ things taken $k$ at a time is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The number of *combinations* (unordered subsets) of $n$ things taken $k$ at a time is

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

The number of combinations is typically referred to as "$n$ choose $k$", and is also written

$$\binom{n}{k} = C(n,k).$$

The *cartesian product* of the sets $A$ and $B$ is

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

More generally, the cartesian product of sets $A_1, \dots, A_n$ is the set of ordered $n$-tuples whose $i^{\text{th}}$ entry is from the set $A_i$.

## 2. Compound Events

Suppose that $S$ is a sample space and $A, B \subset S$. Then $A$ and $B$ are events. We are interested in computing the probability that either of these events occurs, and the probability that both of these events occur.

- $P(A \cup B) = P(A \text{ or } B)$
- $P(A \cap B) = P(A \text{ and } B)$

The events $A \cup B$ and $A \cap B$ are known as *compound events*.

This lesson deals with the first case, and the next lesson deals with the second case.

It may seem that the probability of $A$ or $B$ may be the probability of $A$ plus the probability of $B$, but this is now always the case.

## 3. Motivational Example

**Example 1.** A coin is flipped five times. What is the probability of obtaining at least four heads?

*Discussion.* We realize that "at least four heads" can be translated as "exactly four heads or exactly five heads". How does this OR condition effect our probability computation? □

## 4. Disjoint Events

Let $A$ and $B$ be sets. Recall that the union $A \cup B$ is the set of all elements which are in $A$ OR are in $B$, and that the intersection $A \cap B$ is the set of all elements which are in $A$ AND are in $B$.

Also recall that the *empty set* is the set with no elements. It is denoted $\varnothing$. Then

$$|\varnothing| = 0.$$

We say that $A$ and $B$ are *disjoint* if $A$ and $B$ have no elements in common; that is, if $A \cap B = \varnothing$.

Let $S$ be a probability space and let $A, B \subset S$. We say that $A$ and $B$ are disjoint events if $A \cap B = \varnothing$.

If $A$ and $B$ are disjoint, then

$$|A \cup B| = |A| + |B|.$$

In this case, the probability of $A$ or $B$ is

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B|}{S} = \frac{|A|}{|S|} + \frac{|B|}{|S|} = P(A) + P(B).$$

Disjoint events are sometimes called *mutually exclusive* events.

**Example 1** (continued)**.** A coin is flipped five times. What is the probability of obtaining at least four heads?

*Solution.* Let $F = \{0, 1\}$, where 0 represents tails and 1 represents heads. Then

$$S = F^5 = \{ \text{ ordered quintuples from } F \}.$$

Thus, $|S| = 2^5 = 32$.

Let $A$ be the event of exactly four heads, and let $B$ be the event of exactly five heads. Then $|A| = 5$ and $|B| = 1$. Since these are disjoint events,

$$P(A \text{ or } B) = P(A) + P(B) = \frac{5}{32} + \frac{1}{32} = \frac{6}{32} = 0.1875.$$

□

## 5. Motivational Example Again

**Example 2.** Red cards are those whose suit is hearts or diamonds, number cards are those whose rank is two through ten.

Three cards are dealt at random from a standard deck.

**(a)** What is the probability that all three are red?
**(b)** What is the probability that all three are number cards?
**(c)** What is the probability that all three are red number cards?
**(d)** What is the probability that all three are either red or number cards?

*Discussion.* There are 26 red cards and 36 number cards. Half of the number cards are red, so there are 18 red number cards.

The sample space $S$ is the collection of sets of three cards, and its cardinality is 52 choose 3, so

$$|S| = \binom{52}{3} = 22100.$$

**(a)** The event $A$ is the collection of sets of three red cards, and its cardinality is 26 choose 3, so

$$|A| = \binom{26}{3} = 2600, \quad \text{so} \quad P(A) = \frac{2600}{22100} = 0.118.$$

**(b)** The event $B$ is the collection of sets of three number cards, and its cardinality is 36 choose 3, so

$$|B| = \binom{36}{3} = 7140, \quad \text{so} \quad P(B) = \frac{7140}{22100} = 0.323.$$

**(c)** The event $C$ is the collection of sets of three red number cards, and its cardinality is 18 choose 3, so

$$|C| = \binom{18}{3} = 816, \quad \text{so} \quad P(C) = \frac{816}{22100} = 0.037.$$

**(d)** Since the sets $A$ and $B$ overlap, we cannot simply add the probabilities. We introduce a new mathematical idea.

□

## 6. Inclusion-Exclusion Principle

If two set overlap, how do we compute the cardinality of their union?

Clearly,

$$|A \cup B| \leq |A| + |B|.$$

If $A$ and $B$ are disjoint, this inequality is an equal sign. If they overlap, adding the cardinalities will have counted the overlapping part twice. So, to obtain the correct cardinality, we subtract the overlap from the sum. The overlap is the intersection of the two sets. This gives us the formula

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Dividing through by $S$ gives

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

**Example 2** (continued)**.** Five cards are dealt at random from a standard deck. What is the probability that all three are either red or number cards?

*Solution.* Continue with the notation

- $S$ is the collection of sets of three cards.
- $A$ is the collection of sets of three red cards.
- $B$ is the collection of sets of three number cards.
- $C$ is the collection of sets of three red number cards.
- $D$ is the collection of sets of cards which are either red cards or number cards.

We see that $C = A \cap B$ and $D = A \cup B$. Thus

$$|D| = |A| + |B| - |C|,$$

so

$$P(D) = P(A) + P(B) - P(C) = 0.118 + 0.323 - 0.037 = 0.404.$$

□